DR SNS RAJALAKSHMI COLLEGE OF ARTS AND SCIENCE

(AUTONOMOUS)

COIMBATORE – 641049

21UMA503: LINEAR ALGEBRA

III B.SC. MATHEMATICS

Unit V Questions

Definition: Minimal Polynomial of a matrix.

The Minimal Polynomial of a matrix is defined as the monic polynomial lowest degree satisfied by the satisfied by the matrix.

Remarks

1. The Minimal Polynomial of a matrix divides every polynomial satisfied by the matrix.
2. The Characteristic Polynomial and the the Minimal Polynomial have the same characteristic factors.
   A scalar is an eigen value of a matrix A if and only if it is the root of the minimal polynomial $m_A\ (x)$.

Define the following terms (i) Jordan Block of a matrix (ii) Jordan Canonical Form of a matrix. COV (L- I).

Solution: (i) **Jordan Block**

A matrix is called Jordan Block if **all the diagonal elements of the matrix are the same scalar** and always **1 occurs over the diagonal elements**. The following are the Examples of Jordan Blocks

$$[10] \quad \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix} \quad \begin{bmatrix} 8 & 1 & 0 \\ 0 & 8 & 1 \\ 0 & 0 & 8 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(ii) Jordan Canonical Form of a matrix.

Every Square matrix A whose characteristic polynomial

$$\Psi_A\,(t) = (t - \lambda_1)^{n_1}\,(t - \lambda_2)^{n_2} \dots\dots (t - \lambda_r)^{n_r}$$

And the minimal polynomial

$$m_A\,(t) = (t - \lambda_1)^{m}\,(t - \lambda_2)^{m} \dots\dots (t - \lambda_r)^{m}$$

can be reduced to a block diagonal matrix $\begin{bmatrix} J_{1j} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & J_{rj} \end{bmatrix}$ which is called Jordan Canonical Form, where J_{ij} are Jordan Blocks of the form

$\begin{bmatrix} \lambda_i & 1 & 0 & \ldots & 0 & 0 \\ 0 & \lambda_i & 1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 0 & \lambda_i \end{bmatrix}$, i = 1,2,3 ....r and for each λ_i the corresponding Blocks.

Properties of Jordan Canonical Form

Consider the Jordan Canonical Form $\begin{bmatrix} J_{1j} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & J_{rj} \end{bmatrix}$

(i) There is atleast 1 J_{ij} of order m and all the other J_{ij} *are of order less than m*.
(ii) The sum of the orders of J_{ij} is n_i
(iii)The number of J_{ij} equals the Geometric multiplicity of λ_i.
(iv) The number of J_{ij} of each possible order is uniquely determined by A.

When do we say an Operator T is said to be diagnosable? COV (L- II).

Solution: Let V be an n dimensional vector space. An operator T: $V \longrightarrow V$ is said to be diagonalizable if its matrix representation is diagonalizable.

When do we say a matrix A is said to be diagonalizable? COV (L- II).

Solution: A matrix A is said to be diagonalizable if it is similar to a diagonal matrix D. That is there exists a non – singular matrix P such that $D = P^{-1}AP$.

When do we say a matrix A is said to be orthogonally diagonalizable? COV (L- II).

Solution:

Section B A matrix A is said to be orthogonally diagonalizable if there exists an orthogonal matrix P such that $D = P^{-1}AP$.

1. Find the minimal polynomial of the matrix $\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & 4 \end{pmatrix}$ CO V (L – III).

Solution: Let $A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & 4 \end{bmatrix}$

Then the Characteristic polynomial of A is $|A - xI| = 0$.

$$\begin{vmatrix} 2-x & 1 & 0 & 0 \\ 0 & 2-x & 0 & 0 \\ 0 & 0 & 1-x & 1 \\ 0 & 0 & -2 & 4-x \end{vmatrix} = 0$$

$$\Rightarrow (2-x)\begin{vmatrix} (2-x) & 0 & 0 \\ 0 & (1-x) & 1 \\ 0 & -2 & (4-x) \end{vmatrix} = 0$$

$$\Rightarrow (2-x)\,(2-x)\,\begin{vmatrix} (1-x) & 1 \\ -2 & (4-x) \end{vmatrix} = 0$$

$$\Rightarrow (2-x)^2\,[(1-x)(4-x) + 2] = 0$$

$$\Rightarrow (2-x)^2\,(x^2 - 5x + 6) = 0$$

$$\Rightarrow (x-2)^3\,(x-3) = 0$$

Therefore the Characteristic polynomial of A is $(x-2)^3\,(x-3)$

The possible minimal polynomials are

$f(x) = (x-3)\,(x-2)$ $\quad$ $g(x) = (x-3)\,(x-2)^2$ $\quad$ $h(x) = (x-3)\,(x-2)^3$

$$f(A) = (A-3I)\,(A-2I)$$

$$f(A) = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & -2 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -2 & 2 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq O$$

$$g(A) = (A-3I)\,(A-2I)^2$$

$$g(A) = (A-3I)(A-2I)(A-2I)$$

$$g(A) = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & -2 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -2 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -2 & 2 \end{pmatrix}$$

$$g(A) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = O$$

Therefore minimal polynomial of A is $m_A\,(x) = (x-3)\,(x-2)^2$.

2. Diagonalize the matrix $A = \begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix}$ CO V (L – III).

Solution:

Step I: To find Eigen values

Let the given matrix be $A = \begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix}$. Then the Characteristic equation is $|A - \lambda I| = 0$.

$$\Rightarrow |A - \lambda I| = \left| \begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = \begin{vmatrix} 1-\lambda & 4 \\ 2 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow -(1-\lambda)(1+\lambda) - 8 = 0$$

$$\Rightarrow -(1-\lambda^2) - 8 = 0$$

$$\Rightarrow \lambda^2 = 9 \Rightarrow \lambda = 3, -3 \text{ which are the eigen values of A.}$$

Step 2: To find Eigen vectors and P matrix.

Case (i) Put $\lambda = -3$

$(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$(1-\lambda)x_1 + 4x_2 = 0$

$2x_1 - (1+\lambda)x_2 = 0$

Put $\lambda = -3$ in the above equations we get ,

$\mathbf{4}x_1 + 4x_2 = 0$

$2x_1 + 2x_2 = 0$

$\Rightarrow x_1 + x_2 = 0$

$\Rightarrow x_1 = -x_2$

$\Rightarrow \frac{x_1}{-1} = \frac{x_2}{1}$

Therefore eigen vector corresponding to $\lambda = -3$ is $X_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Case (ii) Put $\lambda = 3$ in the above equations we get,

$-2x_1 + 4x_2 = 0$

$2x_1 - 4x_2 = 0$

$\Rightarrow 2x_1 = 4x_2$

$\Rightarrow \frac{x_1}{4} = \frac{x_2}{2}$

$$\Rightarrow \frac{x_1}{2} = \frac{x_2}{1}$$

Therefore eigen vector corresponding to $\lambda = 3$ is $X_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

These eigen vectors are linearly independent. Therefore $P = [X_1 \ X_2] = \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}$

Step 3: To find P^{-1}. Use the formula $\boldsymbol{P^{-1} = \frac{adjP}{|P|}}$.

Here $\lceil P \rceil = -3$ and adjP = $\begin{pmatrix} +1 & -1 \\ -2 & +(-1) \end{pmatrix}^t = \begin{pmatrix} +1 & -1 \\ -2 & -1 \end{pmatrix}^t = \begin{pmatrix} 1 & -2 \\ -1 & -1 \end{pmatrix}$

$$P^{-1} = \frac{1}{3}\begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix}$$

Step 4: To find $P^{-1}AP$

$$P^{-1}AP = \frac{1}{3}\begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 3 & -6 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} -9 & 0 \\ 0 & 9 \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = D$$

Since Therefore $P^{-1}AP = D$, the given matrix A is diagonalizable.

Section C

1. Find the Characteristic Polynomial and the Minimal Polynomial of the matrix $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{pmatrix}$. COV (L – IV)

Solution: The given matirix is $\boldsymbol{A} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{pmatrix}$. Then the Characteristic equation is

$|A - xI| = 0.$

Therefore $A - xI = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{pmatrix} - \text{x}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2-x & 1 & 0 \\ 0 & 1-x & -1 \\ 0 & 2 & 4-x \end{pmatrix}$

$\Rightarrow \ |A - xI| = 0.$

$$\Rightarrow \begin{vmatrix} 2-x & 1 & 0 \\ 0 & 1-x & -1 \\ 0 & 2 & 4-x \end{vmatrix} = 0.$$

$\Rightarrow (2-x)[(1-x)(4-x)+2] \ = 0.$

$\Rightarrow (2-x)(x^2 - 5x + 6) \ = 0.$

$\Rightarrow (2-x)(x-2)(x-3) = 0.$

$\Rightarrow (x-2)^2(x-3) = 0.$

Therefore the Characteristic polynomial is $(x-2)^2(x-3) = 0$

Also the possible minimal polynomials are $(x-2)(x-3)$ *and* $(x-2)^2(x-3)$

Now Replace x by A we get $(A-2I)(A-3I)$ *and* $(A-2I)^2(A-3I)$

Now, $(A-2I)(A-3I) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \end{pmatrix}\begin{pmatrix} -1 & 1 & 0 \\ 0 & -2 & -1 \\ 0 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq O.$

A does not satisfy the polynomial $(x-2)(x-3)$.

But, By Cayley Hamilton theorem, Every square matrix satisfies its Characteristic equation.

Therefore by above theorem, A satisfies the polynomial $(x-2)^2(x-3)$.

Minimal Polynomial of A is $(x-2)^2(x-3)$.

Hence, the Characteristic Polynomial is also the Minimal Polynomial of A.